Lecture 2 (12/21/2021)

Pf: We can solve the eq (**) by using the method of characteristics

$$h(t,\underset{x}{k},\underset{v}{\xi}) = \int_{-\infty}^{t} e^{-\lambda(t-t') - \int_{t'}^{t} \underbrace{a^{ij}(s)(\xi_i - k_i(s-t))(\xi_j - k_j(s-t))}_{\geq \delta\left(\frac{|k|^2(t-t')^3}{3} + (t-t')^2\xi\cdot k + (t-t')|\xi|^2\right)}} f(t',k,\xi - k(t'-t))\,dt'$$

Young's inequality and Minkowski's ineq.

$t-t'=s$

$$\lambda|h| \leq \lambda\int_0^\infty e^{-\lambda s - \underbrace{\delta(|k|^2 s^3/3 + s^2\xi\cdot k + s\cdot|\xi|^2)}_{\geq 0}} |f(t-s,k,\xi+sk)|\,ds.$$

$$\leq \lambda\int_0^\infty e^{-\lambda s}|f(t-s,k,\xi+sk)|\,ds$$

By using the Minkowski's ineq. $\|\lambda h\|_{L_2} \leq \lambda\int_0^\infty e^{-\lambda s}\underbrace{\|f(t-s,k,\xi+sk)\|_{L_2}}_{\leq \|f\|_{L_2}}$

$$|k|^{2/3}h \leq \int_0^\infty |k|^{2/3} e^{-\tilde{\delta}|k|^2 s^3}|f(t-s,k,\xi+sk)|\,ds = \int_0^\infty e^{-\tilde{\delta}\tau^3}\underbrace{|f(t-|k|^{-2/3}\tau, k, \xi+\tau|k|^{-2/3}k)|}_{=\|f\|_{L_2}}\,d\tau$$

Change of variables $|k|^{2/3}s \to \tau$.

$$\||k|^{2/3}h\|_{L_2} \leq \int_0^\infty e^{-\tilde{\delta}\tau^3}\|f\|_{L_2}\,d\tau \leq N\|f\|_{L_2}. \qquad t\in(-\infty,T)$$

By using the Parseval's identity, we get ① of thm.

Next, we prove the solvability $u_t - a_{ij}(t)D_{ij}u = f$

- Caccioppoli's inequality. (energy inequality)

Lem 2: Suppose $P_0 u + \lambda u = f$ in Q_2. $u\in S_{2,loc}$ ($u, D_v u, D_v^2 u, u_t + v\nabla_x u \in L_{2,loc}$)

Then $\|D_v u\|_{L_2(Q_1)} + \|D_v^2 u\|_{L_2(Q_1)} + \lambda\|u\|_{L_2(Q_1)} \leq N(d,\delta)(\|f\|_{L_2(Q_2)} + \|u\|_{L_2(Q_2)})$.

Pf: $u_t + \overset{x_0 u}{\boxed{v\nabla_x u}} - a_{ij}(t)D_{v_iv_j}u + \lambda u = f$ ① $u\eta^2$ η suitable cutoff function.

$x_i, x_j u$

$$\int u_t u\eta^2 + \int v\nabla_x u\cdot u\eta^2 + \int a_{ij}(t)D_{v_j}u\, D_{v_i}(u\eta^2) + \int\lambda u^2\eta^2$$

$\nabla_x\frac{|u|^2}{2}$. $\qquad \delta|D_v u|^2$.

$$\leq \int fu\eta^2$$

$$\leq \int f^2\eta^2 + \int u^2\eta^2.$$

$$\Rightarrow \int_{Q_1}\lambda u^2 + \int_{Q_1}|D_v u|^2 \leq N\int_{Q_2}(|f|^2 + |u|^2).$$

In the case of the heat eq, we can differentiate the eq in x_j and get the estimate of $D^2 u$.

$$(\partial_v u)_t + v\nabla_x\partial_v u - a_{ij}D_{v_iv_j}\partial_v u + \lambda\partial_v u = \partial_v f - \underbrace{\nabla_x u}_{\partial_v^3 u} \qquad x\sim v^3$$

Correct pf: $\eta \in C_0^\infty(\tilde{Q}_2)$, $\eta \equiv 1$ in $\tilde{Q}_1$.

$(P_0+\lambda)(\eta u) = \underbrace{\eta(P_0+\lambda)u}_{\overset{\parallel}{f\eta}} + u P_0\eta - \underbrace{2a D_v\eta D_v u}$ is satisfied in $(-\infty,0)\times\mathbb{R}^{2d}$

By Thm ①. $\lambda\|\eta u\|_{L_2} + \|D_v^2(\eta u)\|_{L_2} + \cdots \le N(\|f\|_{L_2(Q_2)} + \|u\|_{L_2(Q_2)} + \|D_v u\|_{L_2(Q_2)})$.

$\Rightarrow \underline{\lambda\|u\|_{L_2(Q_1)}} + \underline{\|D_v^2 u\|_{L_2(Q_1)}} \le N(\|f\|_{L_2(Q_2)} + \|u\|_{L_2(Q_2)} + \underbrace{\|D_v u\|_{L_2(Q_2)}})$

$+ \|D_v u\|_{L_2(Q_1)}$.

By using interpolation and the iteration argument.

Lem 3. $\forall \lambda \ge 0$. $(P_0+\lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_2(\mathbb{R}^{1+2d})$. We can get rid of the $D_v u$ term.

pf: If not true. the $\overline{(P_0+\lambda)C_0^\infty}^{L_2}$ is a proper subspace of $L_2(\mathbb{R}^{1+2d})$.

By the Hahn–Banach thm, $\exists\, \ell \in L_2^*(\mathbb{R}^{1+2d})$ $\|\ell\| \ne 0$, but $\ell\left(\overline{(P_0+\lambda)C_0^\infty}^{L_2}\right) = 0$.

But $(L^2)^* = L^2$, $\exists\, u \in L_2(\mathbb{R}^{1+2d})$, $u \not\equiv 0$ s.t. $\int_{\mathbb{R}^{1+2d}} (P_0\zeta + \lambda\zeta)\, u \, dz = 0$ $\forall \zeta \in C_0^\infty(\mathbb{R}^{1+2d})$.

$\Rightarrow \underbrace{P_0^*}_{\text{adjoint operator}} u + \lambda u = 0$ in the sense of distribution

$-\partial_t u - v\cdot\nabla_x u - a_{ij}(t) D_{v_iv_j} u + \lambda u = 0$. Denote $w(t,x,v) = u(-t,-x,v)$.

Then $w_t + \underbrace{v\cdot\nabla_x w} - \underbrace{a_{ij}(t) D_{v_iv_j} w} + \lambda w = 0$. Difficulty $w=0$

We use mollification, $w_\varepsilon(z) = \int w(t, x-\sqrt{\varepsilon}x', v-\varepsilon v')\,\eta(x',v')\,dx'dv'$ (mollification in x and v)

w_ε satisfies $\partial_t w_\varepsilon + v\cdot\nabla_x w_\varepsilon - a_{ij}(t) D_{v_iv_j} w_\varepsilon + \lambda w_\varepsilon = \underbrace{v\nabla_x w_\varepsilon - (v\nabla_x w)_\varepsilon}_{g}$.

$g = \int \varepsilon v' D_x w(t, x-\sqrt{\varepsilon}x', v-\varepsilon v')\,\eta = -\int \sqrt{\varepsilon}\, v' D_{x'} w(t, x-\sqrt{\varepsilon}x', v-\varepsilon v')\,\eta(x',v')\,dx'dv'$

$= \int \sqrt{\varepsilon}\, v' w(t, x-\sqrt{\varepsilon}x', v-\varepsilon v')\, D_{x'}\eta(x',v')\,dx'dv'$

$\|g\|_{L_2} \le N\sqrt{\varepsilon}\,\|w\|_{L_2}$.

By using the Caccioppoli ineq. $\Rightarrow \|D_v w_\varepsilon\|_{L_2(Q_r)} \le N\left(r\|g\|_{L_2(Q_{2r})} + \frac{1}{r}\|w_\varepsilon\|_{L_2(Q_{2r})}\right)$.

$w_\varepsilon \in S_{2,loc}$. $D_v w_\varepsilon$, $D_v^2 w_\varepsilon$, $D_x w_\varepsilon \in L_{2,loc}$. $\Rightarrow \partial_t w_\varepsilon \in L_{2,loc} \Rightarrow w_\varepsilon \in S_{2,loc}$.

$\Rightarrow \|D_v w_\varepsilon\|_{L_2(Q_r)} \le N\underbrace{\left(r\sqrt{\varepsilon} + \frac{1}{r}\right)}_{\text{unif in } \varepsilon}\|w\|_{L_2}$. $\varepsilon \in (0,1)$

$\Rightarrow D_v w \in L_{2,loc}$

Let $\varepsilon \to 0$, $\|D_v w\|_{L_2(Q_r)} \le \frac{N}{r}\|w\|_{L_2}$

Let $r \to \infty$ $\Rightarrow \|D_v w\|_{L_2(\mathbb{R}_0^{1+2d})} = 0$ $\Rightarrow$ w independent of v

$\mathbb{R}_0^{1+2d} = (-\infty,0)\times\mathbb{R}^{2d}$

But $w \in L_2$

$\Rightarrow w \equiv 0$

But u is nonzero, so we get a contradiction.

pf of ② of the thm: Let $T=\infty$ $(-\infty,+\infty)\times\mathbb{R}^{2d}$, $f\in L_2(\mathbb{R}^{1+2d})$.

According to Lem 3. we can find $\{u_k\}\subset C_0^\infty(\mathbb{R}^{1+2d})$ s.t $(P_0+\lambda)u_k=f_k\xrightarrow{L^2} f$ as $k\to\infty$

Then by part ①. $\lambda\|u_k-u_j\|_{L_2}+\cdots+\|D_v^2(u_k-u_j)\|_{L_2}\le N\|f_k-f_j\|_{L_2}\to 0$ as $k,j\to\infty$.

$\{u_k\}$ is a Cauchy sequence in S_2. Let u be the limit., then u is a solution.

If $T<\infty$ we take the zero extension of f when $t>T$. Solve the equation in $\mathbb{R}^{1+2d}$. then restrict the solution to $\mathbb{R}^{1+2d}_T$.

③ Cauchy problem: take the zero extension of f when $t<0$, then solve the equation in $\mathbb{R}^{1+2d}_T$. u. $u|_{\mathbb{R}^{1+2d}_0}$ Satisfies the equation $(P_0+\lambda)u=0$ in $\mathbb{R}^{1+2d}_0$

By the uniqueness of sol, $\Rightarrow$ $u\equiv 0$ when $t<0$

Thus u satisfies the zero initial condition when $t=0$.

Q: $D_x u$, $D_v^3 u$?? $X_0=\partial_t+v D_x$ $X_i=D_{v_i}$ $i=1,\cdots,d$. $D_{x_i}=[D_{v_i},X_0]$

Lem (Caccioppli's ineq in x). $u\in S_{2,loc}$. $(P_0+\lambda)u=0$ in $\boxed{Q_2}$ Then

$\|D_x u\|_{L_2(Q_1)}\le N\|u\|_{L_2(Q_2)}$ No D_x^2 in the equation.

pf: First of all. we may that u has a compact support. Then we mollify u in x, so that $D_x^k u\in L_2$. $(-\Delta_x)^\beta u\in L_2$. $\forall\beta\ge 0$

Let $\phi=\phi_1(t,v)\phi_2(x)$ where ϕ_1 and ϕ_2 are suitable cut off functions

$(P_0+\lambda)(\phi u)=u P_0\phi-2aD_v\phi D_v u$. $(*)$ is satisfied in the whole space.

By the theorem. $\|(-\Delta_x)^{1/3}(\phi u)\|_{L_2}\le N(\|uP_0\phi\|_{L_2}+\|aD_v\phi D_v u\|_{L_2})$.

$\Rightarrow\|\phi_1(-\Delta_x)^{1/3}(\phi_2 u)\|_{L_2}\le N\|u\|_{L_2(Q_2)}$. $(**)$

Acting $(-\Delta_x)^{1/3}$ on $(*)$. $w=(-\Delta_x)^{1/3}(u\phi)$. $\Rightarrow$ $(P_0+\lambda)w=(-\Delta_x)^{1/3}(uP_0\phi)$
$-2aD_v\phi_1 D_v(-\Delta_x)^{1/3}(u\phi_2)$

By the theorem again. $\|(-\Delta_x)^{2/3}u\phi\|_{L_2}=\|(-\Delta_x)^{1/3}w\|_{L_2}$

$\le N(\underbrace{\|(-\Delta_x)^{1/3}uP_0\phi\|_{L_2}}_{\overset{(**)}{\le} N\|u\|_{L_2(Q_2)}}+\|aD_v\phi D_v\underbrace{(-\Delta_x)^{1/3}(u\phi_2)}_{f}\|_{L_2})$

2nd term

$(P_0+\lambda)u\phi_2=v(\nabla_x\phi_2)u$.

$\Rightarrow(P_0+\lambda)f=(-\Delta_x)^{1/3}(v\cdot\nabla_x\phi_2)u$.

2nd term: $\le N\|\underbrace{(-\Delta_x)^{1/3}(u\phi_2)}_{f}\|_{L_2(Q_{3/2})}+N\|(-\Delta_x)^{1/3}(v)(\underbrace{D_x\phi_2}_{\phi_2})u\|_{L_2(Q_{3/2})}$

$\le N\|u\|_{L_2(Q_2)}$.

$\Rightarrow$. $\|u\phi\|_{L_2(Q_1)}+\|\underbrace{(-\Delta_x)^{2/3}}_{4/3}(u\phi)\|_{L_2(Q_1)}\le N\|u\|_{L_2(Q_2)}$.

Interpolation $\|D_x(u\phi)\|_{L_2}\le LHS\le N\|u\|_{L_2(Q_2)}$. $\square$

Nonlocal gradient estimate.

$Q_r(z_0) = \{t_0 - r^2 < t < t_0,\ |v - v_0| < r,\ |x - x_0 - v_0(t - t_0)| < r^3\}$

$Q_{r,R}(z_0) = \{t_0 - r^2 < t < t_0,\ |v - v_0| < r,\ |x - x_0 - v_0(t-t_0)| < R^3\}$ $\quad R \ge r.$

Lem 5: $r \in (0,1)$. $u \in S_{2,loc}$. $(P_0 + \lambda)u = 0$ in $(-1,0) \times \mathbb{R}^d \times B_1$ $\qquad (f)_\Omega =$ average in Ω

Then ① $\|D_x u\|_{L_2(Q_r)} \le N \sum_{k=0}^{\infty} 2^{-k} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_1, 2^k}$ $\times 6$

② $\|D_x u\|_{L_2(Q_r)} \le N \sum_{k=0}^{\infty} 2^{-2k} \left(|(-\Delta_x)^{1/6} u|^2\right)^{1/2}_{Q_1, 2^k}$

Pf: R_x – Riesz transform in x.

$\eta^2 D_x u = \eta^2 R_x (-\Delta_x)^{1/2} u = \eta^2 \underbrace{R_x (-\Delta_x)^{1/6}}_{A} (-\Delta_x)^{1/3} u = \eta A(\eta \underbrace{(-\Delta_x)^{1/3} u}_{w}) - \eta[A, \eta] \underbrace{(-\Delta_x)^{1/3} u}_{w}$

$+$ commutator term.

$w = (-\Delta_x)^{1/3} u$. $(P_0 + \lambda) w = 0$ in $(-1,0) \times \mathbb{R}^d \times B_1$

$(P_0 + \lambda)(\eta w) = \underline{w P_0 \eta - 2a D_v w D_v \eta}$ $\quad$ apply the theorem

$\|(-\Delta_x)^{1/6} (\eta w)\|_{L_2} \le N (\|\eta w\|_{L_2} + \|(-\Delta_x)^{1/3} (\eta w)\|_{L_2}) \le N \|w\|_{L_2(Q_1)}$

interpolation

$\|A(\eta w)\|_{L_2} \le N \|w\|_{L_2(Q_1)}$.

Estimate of the commutator term: $\eta \overbrace{(-\Delta_x)^{1/6} R_x}^{A} w - \overbrace{(-\Delta_x)^{1/6} R_x}^{A} (\eta w)$

Recall. $(-\Delta_x)^{1/6} u = C_d \int \frac{u(x+y) - u(x)}{|y|^{d+1/3}}\, dy$.

Ex: $\underbrace{R_x (-\Delta_x)^{1/6}}_{A} u = \tilde{C}_d \int \frac{y}{|y|^{d+4/3}} (u(x+y) - u(x))\, dy$

Thus $|\text{Comm}| \le \int |w(x+y)| |\eta(x+y) - \eta(x)| \cdot |y|^{-d-1/3}\, dy$ $\qquad |y| \sim 2^{3k}$

$\text{Comm} = \eta(x) \int \frac{y}{|y|^{d+4/3}} (w(x+y) - \cancel{w(x)})\, dy - \int \frac{y}{|y|^{d+4/3}} (w(x+y)\eta(x+y) - \cancel{w(x)\eta(x)})\, dy$

x small. Decompose in y into dyadic shells.

$\|\text{Comm}\|_{L_2} \le N \sum_{k=0}^{\infty} 2^{-k} (w^2)^{1/2}_{Q_1, 2^k}$

Minkowski's ineq.

Lem 6. $P_0 u + \lambda u = 0$ $\quad (-1,0) \times \mathbb{R}^d \times B_1$ Then.

$\boxed{①}$ $\|D_x^l D_v^m u\|_{L_\infty(Q_{1/2})} + \|\partial_t D_x^l D_v^m u\|_{L_\infty(Q_{1/2})} \le N \|u\|_{L_2(Q_1)}$

② $\|D_x^l D_v^{m+1} (-\Delta_x)^{1/6} u\|_{L_\infty(Q_{1/2})} + \|\partial_t \cdots\|_{L_\infty(Q_{1/2})} \le N \|D_v (-\Delta_x)^{1/6} u\|_{L_2(Q_1)}$

$+ N \sum_{k=0}^{\infty} 2^{-2k} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{1,2^k}} + \sqrt{\lambda} \|(-\Delta_x)^{1/6} u\|_{L_2(Q_1)}$

③ $\|D_x^l D_v^{m+2} u\|_{L_\infty(Q_{1/2})} + \|\partial_t \cdots\|_{L_\infty(Q_{1/2})} \le N \|D_v^2 u\|_{L_2(Q_1)} + N \sum_{k=0}^{\infty} 2^{-k} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{1,2^k}}$

$+ N\lambda \|u\|_{L_2(Q_1)}$

Notes

Pf. ① ⇒ ②: $\underbrace{u-(u)_{Q_{3/4}}}_{\text{Still a solution.}}$ $\quad \|D_x^{\ell} D_v^{m+1} u\|_{L_\infty(Q_{1/2})} + \|\partial_t D_x^{\ell} D_v^{m+1} u\|_{L_\infty(Q_{1/2})}$

$\le N \|u-(u)_{Q_{3/4}}\|_{L_2(Q_{3/4})}$

(by Poincaré) $\le N(\|\partial_t u\|_{L_2(Q_{3/4})} + \|D_x u\|_{L_2(Q_{3/4})} + \|D_v u\|_{L_2(Q_{3/4})})$.

$\partial_t u = -v D_x u - \lambda u + a_{ij} D_v^2 u$. $\quad \le N(\|D_x u\|_{L_2(Q_{3/4})} + \|D_v u\|_{L_2(Q_{3/4})}$

$+ \|D_v^2 u\|_{L_2(Q_{3/4})} + \boxed{\lambda \|u\|_{L_2(Q_{3/4})}})$

$\le \sqrt{\lambda} \|u\|_{L_2(Q_{7/8})}$

Caccioppoli.

$\sqrt{\lambda}\|u\|_{L_2} \le \|u\|$.

$P_0 u + \lambda u = 0$ $\quad P_0 D_v u + \lambda D_v u = D_x u$.

$\|D_v^2 u\|_{L_2(Q_{3/4})} \le \|D_v u\|_{L_2(Q_{7/8})} + \|D_x u\|_{L_2(Q_{7/8})}$.

$\le N(\|D_x u\|_{L_2(Q_{7/8})} + \|D_v u\|_{L_2(Q_{7/8})})$.

Replacing u by $(-\Delta_x)^{1/6} u$.

① ⇒ ③. Consider $w = u - (u)_{Q_{3/4}} - v \cdot (D_v u)_{Q_{3/4}}$ still satisfies the equation

$\|D_x^{\ell} D_v^{m+2} u\|_{L_\infty(Q_{1/2})} + \|\partial_t \cdots\|_{L_\infty(Q_{1/2})} \le \boxed{N \|w\|_{L_2(Q_{3/4})}}$

$\| \; D_x^{\ell} D_v^{m+2} w$.

$\le N(\|\partial_t u\|_{L_2(Q_{3/4})} + \|D_v u - (D_v u)_{Q_{3/4}}\|_{L_2(Q_{3/4})} + \|D_x u\|_{L_2(Q_{3/4})})$.

$\le N(\|\partial_t u\|_{L_2(Q_{3/4})} + \boxed{\|\partial_t D_v u\|_{L_2(\cdot\cdot)}} + \|D_v^2 u\|_{L_2(\cdot\cdot)} + \|D_x D_v u\|_{L_2} + \|D_x u\|_{L_2})$.

$\le (\|D_x u\| + \|D_v^2 u\|_{L_2} + \lambda \|u\|_{L_2})$. $\qquad + \|D_v^3 u\| + \lambda \|D_v u\|)$

$\partial_t u = \cdot\cdot v \cdot \nabla_x u + \cdot\cdot D_v^2 u + \cdot\cdot \lambda u$ $\qquad \partial_t D_v u = \cdot\cdot D_x u + v \cdot D_x D_v u + D_v^3 u + \lambda D_v u$.

$\le D_v u \qquad \le D_v u + D_x u$

$\Rightarrow \|w\|_{L_2(Q_{3/4})} \le N(\underbrace{\|D_x u\|_{L_2(Q_{7/8})}}_{\text{By Lem 5}} + \|D_v^2 u\|_{L_2(Q_{7/8})} + \lambda \|u\|_{L_2(Q_{7/8})})$

Application:

Landau estimate with specular boundary